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Critical dynamics of the kinetic Potts model on some fractals

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Abstract. The critical dynamics of the kinetic Potts model on Koch curves and regular fractals is studied by means of the exact time-dependent renormalization-group method. Different critical dynamics are found on these two families of fractals. It is shown that the value of the dynamic critical exponent z depends on both the Potts dimensionality q and the transition rates asymmetry coefficient α. For Koch curves the scaling law of the dynamics exponent $z = D_f + f(q, \alpha)/\nu$, while for regular fractals $z = D_f + 2f(q, \alpha)/\nu$, where $f(q, \alpha)$ characterizes the dependence of the dynamics exponent z on Potts dimensionality q and the transition rates asymmetry coefficient α, and ν is the static exponent of the correlation length.

1. Introduction

An understanding of the critical dynamics of the Potts model is of great importance in studies of various dynamical phenomena such as the Snoek effect [1], liquid-glass transition [2] and Potts glass theory [3]. Compared to the large body of knowledge that deals with the static critical phenomena of the q-state Potts model (see, e.g., [4, 5]), our understanding of the dynamics near the critical point of the Potts model is rather limited. In the light of the achievements of the renormalization-group (RG) method in understanding both static critical phenomena (see, e.g., [6]) and critical dynamics of the kinetic Ising model [7-9], there has been much interest in extending the RG method to the critical dynamics of the Potts model. Maybe the first application of the RG method to the critical dynamics of the Potts model was suggested by Weir and Kosterlitz [10]. They have used the time-dependent real space renormalization group (TDRG) approach introduced by Achiam [7] for dynamical problems of the $q = 2^n$ one-dimensional (1D) Potts model. The dynamic critical exponent z was found to be 2, independent of q for their choice of the transition rates. Similar to the Ising model, the Potts model does not possess 'intrinsic' dynamics. Up to now, most TDRG studies of the critical dynamics have been limited to the simplest Glauber model [11] or generalized Glauber model. The crucial point in the Glauber analysis is the construction of the master equation and the choice of the transition rates in the master equation. Those transition rates have to obey certain conditions, out of which the detailed balance is the most important one. In the Ising case any choice of the transition rates fulfilling the detailed balance leads to the same value of the dynamics exponent z, while for the $q > 2$ Potts model, various values of z have been obtained depending on the choice of the transition rates [10, 12, 13]. Quite recently, Zaluska-Kotur and Turski [14] have extended the TDRG to study the critical dynamics of the 1D Potts model with a very broad class of transition rates. It is revealed that the dynamic critical exponent z may or may not depend on q for different choices of the transition rates.

Until now, there has not been any study of the critical dynamics of the Potts model on a fractal. The critical-dynamics behaviour of spin models on fractals is interesting since the simple $q=2$ Potts model (Ising model) has been shown to possess different critical-dynamics behaviours on different fractals [8, 9, 15]. For instance, the scaling law of the dynamic critical exponent $z=D_f+1/\nu$ is found on the Koch curves [9] whereas on regular fractals one has $z=D_f+2/\nu$. Here D_f and ν are the fractal dimension and the static correlation exponent, respectively. In this paper we study the critical dynamics of the Potts model with any q and a broad class of transition rates [14] on some finitely ramified fractals using the TDRG approach. We also expect that our studies would shed some light on the critical dynamics of random systems such as a diffusion-limited-aggregation (DLA) cluster.

The paper is organized as follows. In section 2 the kinetic Potts model is characterized and the TDRG on the Potts model is briefly discussed. The Koch curves are discussed in section 3. The dynamics on the regular fractal for the DLA cluster are presented in section 4. Finally we conclude and discuss in section 5.

2. Kinetic Potts model and formulation of the TDRG method

The Potts system can be described by the Hamiltonian

$$\beta H = -K \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{1}$$

where the Potts spins σ_i are placed at every junction of the fractal and assume the value $\sigma_\alpha = -q/2, -q/2+1, \ldots, -1, 1, \ldots, q/2$ and $\sigma_\alpha = -(q-1)/2, -(q-1)/2+1, \ldots, -1, 0, 1, \ldots, (q-1)/2$ for even and odd q, respectively. The summation in (1) is restricted to nearest neighbours and $\sigma_\alpha \sigma_\beta = q\delta_{\sigma_\alpha \sigma_\beta} - 1$ with $\delta_{\sigma_\alpha \sigma_\beta}$ being the Kronecker delta function. The Glauber-like master equation governing this model dynamics is

$$-\tau_0 \frac{\partial}{\partial t} P(\{\sigma\}; t) = \sum_i \sum_{\tilde{\sigma}_i \neq \sigma_i} [1 - p(\sigma_i \to \tilde{\sigma}_i)] W(\sigma_i \to \tilde{\sigma}_i) P(\{\sigma\}; t) \tag{2}$$

where $P(\{\sigma\}; t)$ and $W(\sigma_i \to \tilde{\sigma}_i)$ are the probability distribution and the transition rates, respectively, and $p(\sigma_i \to \tilde{\sigma}_i)$ is a spin-flip operator:

$$p(\sigma_i \to \tilde{\sigma}_i) f(\sigma_1, \ldots, \sigma_i, \ldots, \sigma_N) = f(\sigma_1, \ldots, \tilde{\sigma}_i, \ldots, \sigma_N).$$

The transition rates $W(\sigma_i \to \tilde{\sigma}_i)$ should satisfy the detailed balance condition

$$[1 - p(\sigma_i \to \tilde{\sigma}_i)] W(\sigma_i \to \tilde{\sigma}_i) P_e(\{\sigma\}) = 0 \tag{3}$$

where $P_e(\{\sigma\})$ characterizes the equilibrium state

$$P_e = \frac{e^{-\beta H}}{\mathrm{Tr}[e^{-\beta H}]}.$$

The detailed balance condition does not determine W_i uniquely. Following Zaluska-Kotur and Turski [14], we take

$$W(\sigma_i \to \tilde{\sigma}_i) = \frac{1}{P_e} (P_e p(\sigma_i \to \tilde{\sigma}_i) P_e)^\alpha \tag{4}$$

where the real parameter $\alpha \in [0, 1]$ is the spin-flip asymmetry coefficient [14]. It contains additional information about the internal dynamical property of the Potts model. With (4), the master equation (2) can be rewritten as

$$-\tau_0 \frac{\partial}{\partial t} P_e \phi(\{\sigma\}; t) = \sum_i \sum_{\tilde{\sigma}_i \neq \sigma_i} P_e W(\sigma_i \to \tilde{\sigma}_i)[1 - p(\sigma_i \to \tilde{\sigma}_i)]\phi(\{\sigma\}; t) \qquad (5)$$

where $\phi(\{\sigma\}; t) = P(\{\sigma\}; t)/P_e$ measures the deviation from equilibrium. We limit ourselves to the study of the relaxation of an infinitely small magnetic-like perturbation from equilibrium, thus

$$\phi = 1 + \sum_r h_r \sum_i \sigma_{i,r} \qquad (6)$$

where r distinguishes between points which may have different coordination number.

According to Achiam [8], the TDRG method consists of two steps. The first step is a renormalization of the space by a factor b^{D_f} using the decimation procedure. Multiplying (5) by $T(\mu, \sigma)$, given by

$$T(\mu, \sigma) = \prod_{i=1}^{N} \delta(\mu_i - \sigma_i) \qquad (7)$$

where σ_i are the spins at the edges of the generators on Koch curves or at seed points on regular fractals, and taking a trace over all spins, we obtain

$$-\tau_0 \frac{\partial}{\partial t} \sum_{\{\sigma\}} T(\mu, \sigma) P_e \phi(\{\sigma\}; t) = \sum_{\{\sigma\}} T(\mu, \sigma) \sum_i L_i \phi(\{\sigma\}; t) \qquad (8)$$

where

$$L_i = \sum_{\tilde{\sigma}_i \neq \sigma_i} P_e W(\sigma_i \to \tilde{\sigma}_i)[1 - p(\sigma_i \to \tilde{\sigma}_i)].$$

The left-hand side of (8) is nothing other than the standard static RG transformation [6], which transforms $P(\{\sigma\}; t) = P(K, \{h_r\}; t)$ into $P'(K', \{h'_r\}; t)$, where K and $\{h_r\}$ are interaction and field parameters, respectively. In the parameter space (K, h) the RG transformation is described by the recursion relations

$$K' = RK \qquad h' = \Lambda h. \qquad (9)$$

The RG transformation of the right-hand side of (8) transforms the Liouville operator L and the equilibrium probability distribution P_e to L' and P'_e and results in a transformation in the invariant subspace:

$$h_d = \Omega h. \qquad (10)$$

Then as the second stage, by representing h_d in terms of h' and rescaling the time by a factor b^z, $\tau'_0 = b^z \tau_0$, equation (8) restores the form of equation (2). The dynamic critical exponent z can be obtained by

$$\omega/\lambda = b^{-z} \qquad (11)$$

where ω and λ are the largest eigenvalues of Ω and Λ, respectively.

3. Koch curves

3.1. Non-branching Koch curves

The non-branching Koch curves [16] (NBKC) such as in figure 1 are homogeneous curves with a finite ramification number $r \equiv 2$. As far as the magnetic properties are

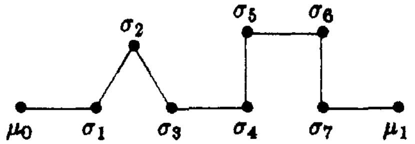

Figure 1. The generator of a simple example of non-branching Koch curves with $b=5$, $l=8$, and $D_f=\ln 8/\ln 5$.

concerned, they are linear 1D chains. The dynamic exponent is obtainable from the Glauber solution. Here we use the TDRG approach to illustrate the formulation of the TDRG method on the q-state Potts model.

There are two terms $\Sigma_{\{\sigma\}}T(\mu,\sigma)P_e$ and $\Sigma_{\{\sigma\}}T(\mu,\sigma)P_e h\,\Sigma_i\sigma_i$ on the left-hand side of (8). Under the RG, we transform the nth stage to the $(n-1)$th stage by tracing over the $(l-1)$ internal spins of the generator. The transfer matrix

$$t(K)=\exp(K\sigma_i\sigma_{i+1}) \tag{12}$$

is transformed to $t'(K')=\exp(K'\mu_j\mu_{j+1})$ by the following relations:

$$At'(K')=[t(K)]^l. \tag{13}$$

Therefore for the first term on the left-hand side of (8) we obtain

$$\sum_{\{\sigma\}} T(\mu,\sigma)P_e=\prod_i A\exp(K'\mu_i\mu_j). \tag{14}$$

The recursion relation of the RG transformation can be obtained from (13)

$$A(x')^{q-1}=\frac{(a+qb)^l}{q}+\frac{q-1}{q}a^l \qquad A(x')^{-1}=\frac{(a+qb)^l}{q}-\frac{a^l}{q} \tag{15}$$

where $a=x^{q-1}-x^{-1}$, $b=x^{-1}$ with $x=e^K$ and $x'=e^{K'}$. Note that when $q=2$ we produce the familiar relation $\tanh K'=(\tanh K)^l$ and $A=2^{l-1}(\cosh K)^l/\cosh K'$. The recursion relation (15) has stable fixed point $x^*=1$ and unstable fixed point $x^*=\infty$.

For the second term on the left-hand side of (8), the renormalized field near the unstable fixed point $x^*=\infty$ is

$$h'=lh=b^{D_f}h. \tag{16}$$

This relation can be obtained by just counting how many spins contribute to h, as shown below.

Under the RG, we transform the nth stage fractal to $(n-1)$th stage by tracing over $l-1$ internal spins, as shown in figure 1 by $\sigma_1,\ldots,\sigma_{l-1}$; for $l=8$, therefore, we have

$$\begin{aligned}\sum_{\{\sigma\}} T(\mu,\sigma)P_e h\sum_{i=1}^{l-1}\sigma_i &\\ &=\frac{hP_e'}{A\exp(K'\mu_0\mu_1)}\sum_{j_1,\ldots,j_{l-1}} t_{\mu_0,j_1}t_{j_1,j_2}\cdots t_{j_{l-1},\mu_1}\sum_{i=1}^{l-1} j_i\\ &=\frac{hP_e'}{A\exp(K'\mu_0\mu_1)}\left((l-1)\mu_0 a^l\delta_{\mu_0\mu_1}+(\mu_0+\mu_1)\frac{a}{b}\sum_{i=0}^{l-2}C_l^i a^i b^{l-i}\right)\\ &=\frac{l-1}{2}(\mu_0+\mu_1)hP_e'\end{aligned} \tag{17}$$

where $C_l^i=l!/[(l-i)!i!]$. In the last step we have made use of the recursion relation (15) and the fact that near the critical point we have $x\to\infty$ and thus retained only the most dominant terms in the summation.

With the use of (17), the recursion relation for the field parameter (16) can be easily obtained.

Now we turn to the right-hand side of (8), which is the sum of the term $\Sigma_{\tilde{\sigma}_i \neq \sigma_i} h(\sigma_i - \tilde{\sigma}_i)P_e W(\sigma_i \to \tilde{\sigma}_i)$. Due to the choice of $W(\sigma_i \to \tilde{\sigma}_i)$ of (4), it is easy to show that

$$\sum_{\sigma_i} \sum_{\tilde{\sigma}_i \neq \sigma_i} (\sigma_i - \tilde{\sigma}_i) h P_e W(\sigma_i \to \tilde{\sigma}_i) = 0.$$

Thus after the trace is performed, only the term

$$\sum_{\tilde{\mu}_i \neq \mu_i} (\mu_i - \tilde{\mu}_i) h P_e W(\mu_i \to \tilde{\mu}_i)$$

survives. Let us concentrate our attention on the generator from μ_0 to μ_1 as shown in figure 1. Paying attention to our choice of $W(\mu_0 \to \tilde{\mu}_0)$, we note that if we have

$$\sum_{\sigma_1 \ldots \sigma_{l-1}} \exp[K(\sigma_1\sigma_2 + \ldots + \sigma_{l-1}\mu_1)] \exp[\alpha K(\mu_0\sigma_1 + \tilde{\mu}_0\sigma_1)]$$
$$= AD \exp[\alpha K'(\mu_0\mu_1 + \tilde{\mu}_0\mu_1)] \tag{18}$$

it is not difficult to derive

$$\sum_{\tilde{\mu}_0 \neq \mu_0} T(\mu, \delta)(\mu_0 - \tilde{\mu}_0) h P_e W(\mu_0 \to \tilde{\mu}_0) = \sum_{\tilde{\mu}_0 \neq \mu_0} D^2(\mu_0 - \tilde{\mu}_0) h P'_e W'(\mu_0 \to \tilde{\mu}_0) \tag{19}$$

where the index 2 of D comes from the coordination number $r = 2$. The formula (19) implies

$$h_d = D^2 h. \tag{20}$$

In the matrix form, (18) can be rewritten as

$$AD\tilde{t}' = (t)^{l-1}\tilde{t} \tag{21}$$

where

$$\tilde{t}_{\sigma_1\sigma_2} = \exp[\alpha K(\sigma_1\sigma_2 + \sigma_1\tilde{\sigma}_2)]. \tag{22}$$

For $q = 2$, with

$$\tilde{t} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(21) gives directly $D = \cosh K'/\cosh K$. For general $q > 2$, it is in fact impossible to satisfy (21) for each matrix element with a single factor D. Fortunately, paying attention to the fact that the system has the unstable fixed point $x^* = \infty$, which corresponds to the zero critical temperature, and taking into account the summation over $\tilde{\mu}_0$ in (19), near the critical point, one can calculate the factor D by just making comparison between the most dominant terms in both sides of (21), as has been done in [14]. This approach is believed to be tenable only for the system which has a zero critical temperature. So for q not smaller than 2 and near the critical point, we have

$$AD(x')^{\alpha(q-2)} = x^{(l-1)(q-1)}x^{\alpha(q-2)} \tag{23}$$

which yields

$$D = l^{-f(q,\alpha)/2} \tag{24}$$

with

$$f(q, \alpha) = \frac{2[(q-1) - \alpha(q-2)]}{q}. \tag{25}$$

By $b^{-z}=\omega_{\max}/\lambda_{\max}$, we obtain the dynamic critical exponent

$$z=D_f+\frac{f(q,\alpha)}{\nu} \tag{26}$$

where $\nu=\ln b/\ln l$ is the static exponent of the correlation length. Thus we have found that the dynamics exponent z depends on both the Potts dimensionality and the transition rates asymmetry coefficient α. For the Ising case, with $q=2$, we have $z=D_f+1/\nu$, which reproduces the result of [9].

3.2. Modified non-branching Koch curves

Now we turn to the discussion of the modified NBKC [9, 16]. The idea was to decorate a NBKC with non-iterative bonds and thus to have a fractal with a less trivial relationship to the 1D problem. An example of such a curve is shown in figure 2. As in [9], we assume that the two branching points σ_{a1} and σ_{b1} on the two sides of the non-iterative bond K_0 are linked by a chain of l iterative bonds and there are $2m$ spins in the two external sides of σ_{a1} and σ_{b1}. Similarly to NBKC, we have the following recursion relation:

$$A_1t_1=t^l \qquad At'=t^mA_1t_ct^m \tag{27}$$

which yields

$$A_1x_1^{q-1}=(a+qb)^l/q+(q-1)a^l/q \qquad A_1x_1^{-1}=(a+qb)^l/q-a^l/q \tag{28}$$

$$\begin{aligned}A(x')^{q-1}/A_1&=(a+qb)^{2m}(a_c+qb_c)/q+(q-1)a^{2m}a_c/q\\ A(x')^{-1}/A_1&=(a+qb)^{2m}(a_c+qb_c)/q-a^{2m}a_c/q\end{aligned} \tag{29}$$

where

$$a_c=x_c^{q-1}-x_c^{-1} \qquad b_c=x_c^{-1} \tag{30}$$

with $x_c=x_1x_0$. The recursion relations have a non-trivial fixed point $x^*=\infty$ only if $x_0=\infty$. The fixed point has a critical exponent $\nu^{-1}=\ln(2m)/\ln b$. In a way similar to that in the case of NBKC, by a tedious matrix manipulation and retaining only the most dominant terms, near the fixed point one has

$$\sum_{\{\sigma\}}T(\mu,\sigma)P_eh\left(\sum_{i=1}^{l-1}\sigma_i+\sum_{i=1}^{m}\sigma_{ai}+\sum_{i=1}^{m}\sigma_{bi}\right)=\left(\frac{l-1}{2}+m\right)(\mu_0+\mu_1)hP'_e. \tag{31}$$

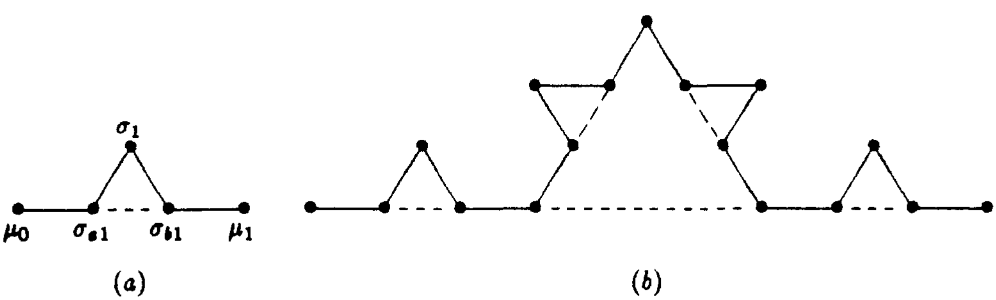

Figure 2. The first two stages of modified non-branching Koch curves with $l=2$ and $m=1$. The solid lines represent iterative bond K and broken lines non-iterative bonds K_0.

With (31) it is easy to obtain

$$h' = \left[1 + 2\left(\frac{l-1}{2} + m\right)\right] h = (2m + l)h = b^{D_f} h. \tag{32}$$

The RG transformation of the right-hand side of (8) can also be performed similarly to the case of NBKC. We can directly write the recursion relation for matrix $\tilde{t}$ as follows

$$AD\tilde{t}' = t^m t_c A_1 t^{m-1} \tilde{t} \tag{33}$$

where the transfer matrices t and $\tilde{t}$ are given in (12) and (22), respectively. Near the zero critical temperature, D can be calculated by comparing the most dominant terms on the two sides of (33), which gives the relation

$$AD(x')^{\alpha(q-2)} = x^{(q-1)m} x_c^{q-1} A_1 x^{(q-1)(m-1)} x^{\alpha(q-2)}. \tag{34}$$

The above formula, together with the recursion relations (28) and (29) provides

$$D = (2m)^{-f(q,\alpha)/2}. \tag{35}$$

Thus the dynamic critical z is

$$z = D_f + \frac{f(q, \alpha)}{\nu}. \tag{36}$$

3.3. Branching Koch curves

For branching Koch curves (BKC), the non-equilibrium Hamiltonian must be parametrized using the interaction K and a set of fields h_r where r is the coordination number. As examples, in figure 3 we show some generators of BKC. The field parameters for the BKC in figure 3(a) should be h_2 and h_3, whereas in figure 3(b) they are h_2 and h_4. The numbers m, n, r, l are defined in figure 3. The RG transformation of the equilibrium probability distribution P_e can be given by the recursion relation for the transfer matrix. For BKC as shown in figure 3(a), we have

$$A_1 t_1 = (t)^l \qquad A_2 t_2 = (t)^r \qquad At' = t^n A_1 A_2 t_c t^m \tag{37}$$

or

$$\begin{aligned} A(x')^{q-1}/(A_1 A_2) &= (a + qb)^{m+n}(a_c + qb_c)/q + (q-1)a^{m+n} a_c/q \\ A(x')^{-1}/(A_1 A_2) &= (a + qb)^{m+n}(a_c + qb_c)/q - a^{m+n} a_c/q \end{aligned} \tag{38}$$

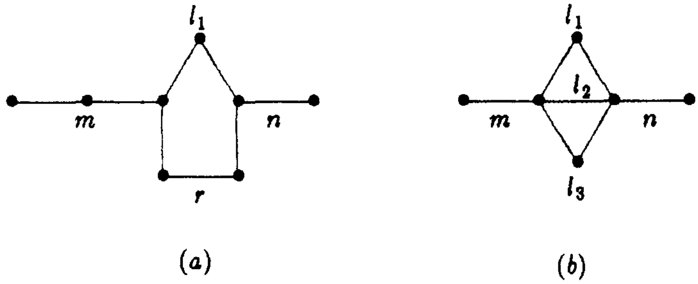

Figure 3. Two examples of the generators of branching Koch curves. The numbers of bonds in different parts of the generators are denoted by the letters in the figure.

where a_c and b_c are given by (30) with $x_c = x_1 x_2$. The recursion relations for x_1, x_2, A_1, A_2 are similar to (28). A direct but tedious matrix manipulation, similar to that in the case of the modified NBKC, shows that the recursion relations for h_2 and h_3 near the unstable fixed point $x^* = \infty$ are simply

$$\begin{aligned} h_2' &= [m-1+(l-1)+(r-1)+m-1+1]h_2+2h_3 \\ h_3' &= \tfrac{3}{2}(2m+r+l-4)h_2+4h_3 \end{aligned} \tag{39}$$

which yields the largest eigenvalue for the transformation matrix Λ

$$\lambda_{\max} = 2m+r+l = b^{D_f}. \tag{40}$$

Note that in the recursion relation for h_3, we must assume a symmetrical generator with $m = n$. If $m \neq n$, one has to either distinguish between the different h_3 according to the symmetry of the vertex or average the contributions. The latter treatment may result in an approximation.

The RG transformation of the right-hand side is straightforward by the renormalization relation for the transfer matrix $\tilde{t}$

$$DA\tilde{t}' = t^m t_c A_1 A_2 t^{m-1} \tilde{t}. \tag{41}$$

Near the fixed point, D can be calculated from the relation

$$DA(x')^{\alpha(q-2)} = x^{m(q-1)} x_c^{q-1} A_1 A_2 x^{(q-1)(m-1)} x^{\alpha(q-2)}. \tag{42}$$

For q not smaller than 2 we have

$$D = \left(\frac{1}{2m}\right)^{f(q,\alpha)/2} \tag{43}$$

which implies

$$h_2' = D^2 h_2 \qquad h_3' = D^3 h_3. \tag{44}$$

The largest eigenvalue of the matrix Ω is D^2, thus one deduces

$$z = D_f + \frac{f(q,\alpha)}{\nu} \tag{45}$$

with $\nu = \ln b/\ln(2m)$ being the static correlation exponent. Therefore, for Koch curves we have the same expression for the dynamics exponent z. The different critical-dynamics behaviours result from the static correlation exponent ν. Indeed, of these three classes of fractals, the modified NBKC will relax the fastest, while the NBKC is the slowest, as in the dynamical Ising case.

4. Regular fractal

In 1981, Witten and Sander [17] proposed the diffusion-limited aggregation (DLA) model by which many phenomena can be described. Later, Christon and Stinchcombe [18] proposed a family of regular models. It is shown that these simple regular fractals retain some of the features of random branching DLA clusters and thus can serve as a good model for the DLA clusters. In this section we study the kinetic Potts model on these regular fractals. We may expect that the study of the critical dynamics on these regular fractals may give some insight into the critical dynamics of random DLA clusters.

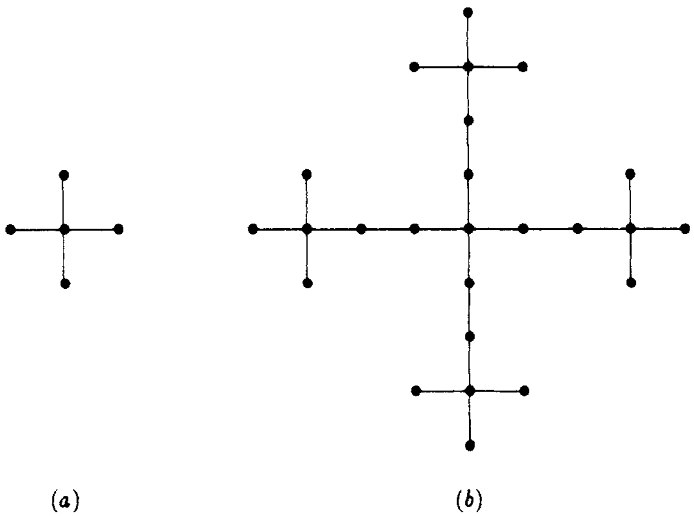

Figure 4. The simplest regular fractal model for DLA with the fractal dimension $D_f = \ln 5/\ln 3$.

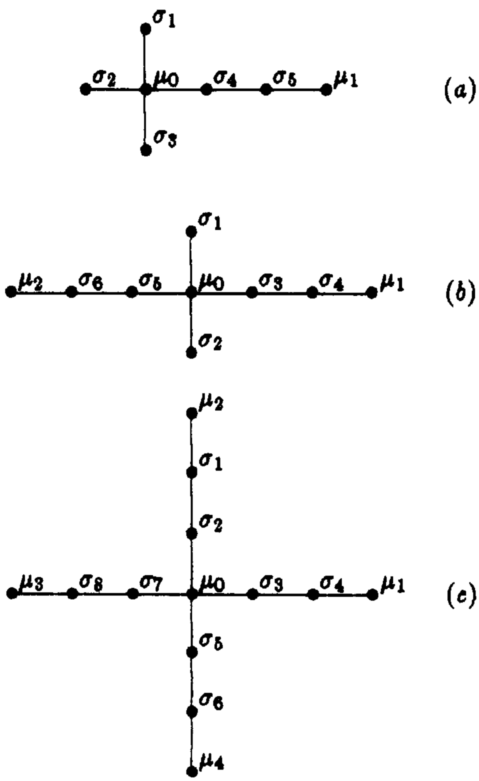

Figure 5. The three typical structures of the simplest regular fractal model for DLA shown in figure 4.

To begin, we focus our attention on a regular fractal as shown in figure 4. The fractal has three kinds of points with different coordination number, ϕ, therefore, is determined by three field parameters h_1, h_2 and h_4 corresponding points with coordination number 1, 2 and 4, respectively. Although the fractal has three typical structures shown in figure 5. These three typical structures produce the same recursion relation for the interaction parameter K. Thus we need only one interaction parameter K and three field parameters h_1, h_2 and h_4 to span the invariant subspace. The three different basic structures with different numbers of dangling ends only produce different renormalization coefficients and thus affect the free energy of the system but not the critical behaviour. After decimation, which eliminates all points except the seed points, we obtain the recursion relations for the transfer matrix for the basic structures with zero, two and three dangling ends, respectively:

$$At' = t^3 \qquad AB^2 t' = t^3 \left(\sum_\sigma e^{\mu_0 \sigma}\right)^2 \qquad AB^3 t' = t^3 \left(\sum_\sigma e^{\mu_0 \sigma}\right)^3. \tag{46}$$

Equation (46) implies

$$\begin{aligned} A(x')^{q-1} &= (a+qb)^3/q + (q-1)a^3/q \\ A(x')^{-1} &= (a+bq)^3/q - a^3/q \end{aligned} \tag{47}$$

and $B = x^{q-1} + (q-1)x^{-1}$. Therefore, for the first term on the left-hand side of (8), we have

$$\sum_{\{\sigma\}} T(\mu, \sigma) P_e = \prod_i A_i B_i \exp(K' \mu_i \mu_j)$$

with B_i being 1, B^2 or B^3 depending on the number of the dangling ends at point i. The recursion relation (47) has a stable fixed point $x^* = 1$ and unstable fixed point $x^* = \infty$.

Before calculating the second term on the left-hand side of (8)

$$\sum_{\{\sigma\}} T(\mu, \sigma) P_e \sum_r h_r \sum_i \sigma_{i,r}$$

we would like to derive some useful relations. For a basic structure with three dangling ends as shown in figure 5(a), it is not difficult to derive

$$\begin{aligned} \sum_{\{\sigma\}} T(\mu, \sigma) P_e h_1 \sigma_1 &= C_1 \mu_0 h_1 P'_e \\ \sum_{\{\sigma\}} T(\mu, \sigma) P_e h_2 (\sigma_4 + \sigma_5) &= \tfrac{1}{2} C_2 (\mu_0 + \mu_1) h_2 P'_e \end{aligned} \tag{48}$$

where

$$C_1 = (x^{q-1} - x^{-1})/B$$

$$C_2 = \frac{a^3 \delta_{\mu_0, \mu_1} + 3a^2 b + ab^2}{A \exp[K'(q\delta_{\mu_0,\mu_1} - 1)]}.$$

Near the fixed point $x^* = \infty$, we have $C_1 = C_2 = 1$.

Equation (48), together with the similar relations for the basic structure with two dangling ends, provides the renormalized fields at the critical point

$$h'_1 = 3h_1 + h_2 + h_4 \qquad h'_2 = 2h_1 + 2h_2 + h_4 \qquad h'_4 = 4h_2 + h_4. \tag{49}$$

Thus the transformation matrix of the field parameters has the largest eigenvalue $\lambda_{\max} = 5 = b^{D_f}$.

Similar to section 3, under RG transformations we have the renormalized transfer matrix $\tilde{t}$

$$
\begin{aligned}
&AD_0\tilde{t}' = t^2\tilde{t}\\
&AD_2B^2\tilde{t}' = \left(\sum_\sigma \exp[\alpha K(\mu_0\sigma + \tilde{\mu}_0\sigma)]\right)^2 t^2\tilde{t}\\
&AD_3B^3\tilde{t}' = \left(\sum_\sigma \exp[\alpha K(\mu_0\sigma + \tilde{\mu}_0\sigma)]\right)^3 t^2\tilde{t}
\end{aligned}
\tag{50}
$$

for the three typical configurations with 0, 2 and 3 dangling ends, respectively. Near the fixed point $x^* = \infty$, D_0, D_2 and D_3 are given by

$$
\begin{aligned}
&D_0 = (3)^{-f(q,\alpha)/2} \qquad D_2 = 4(3)^{-f(q,\alpha)/2}(x)^{-2qf(q,\alpha)}\\
&D_3 = 8(3)^{-f(q,\alpha)/2}(x)^{-3qf(q,\alpha)}.
\end{aligned}
\tag{51}
$$

Thus

$$
\begin{pmatrix} h_1' \\ h_2' \\ h_4' \end{pmatrix} = \begin{pmatrix} 0 & 0 & D_3 \\ 0 & 0 & D_2^2 \\ 0 & 0 & D_0^4 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_4 \end{pmatrix}. \tag{52}
$$

The largest eigenvalue for the transformation matrix Ω is $\omega_{\max} = D_0^4$, therefore the dynamic critical exponent is

$$
z = D_f + \frac{2f(q, \alpha)}{\nu} \tag{53}
$$

with $\nu = 1$ from the recursion relation (47).

For a more complex regular model as shown in figure 6, noting that the dangling ends do not affect the recursion relation for the interaction parameter, we have, analogous to (46), the renormalization equation

$$
At' = t^b. \tag{54}
$$

The above recursion relation has the non-trivial fixed point $x^* = \infty$. Near the fixed point, by a tedious matrix manipulation and retaining only the most dominant terms analogously to section 3, the following recursion relation for the field parameters can be deduced:

$$
h_i' = \sum_j \Lambda_{i,j} h_j \tag{55}
$$

with $\Lambda_{i,j} \geqslant 0$ and $\Sigma_j \Lambda_{i,j} = N$. Here $N = b^{D_f}$ is the number of points in the generator. According to Perrons-Frobenius theorem (see, e.g., [19, 20]), the matrix Λ has the largest eigenvalue $\lambda_{\max} = b^{D_f}$. In order to calculate $\omega_{\max}$, we should pay attention to the fact that the seed points which survive after decimation have the coordination number $r = 4$. So we have

$$
\begin{pmatrix} h_1' \\ h_2' \\ h_4' \end{pmatrix} = \begin{pmatrix} 0 & 0 & \tilde{D}_1 \\ 0 & 0 & \tilde{D}_2 \\ 0 & 0 & \tilde{D}_4 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_4 \end{pmatrix}. \tag{56}
$$

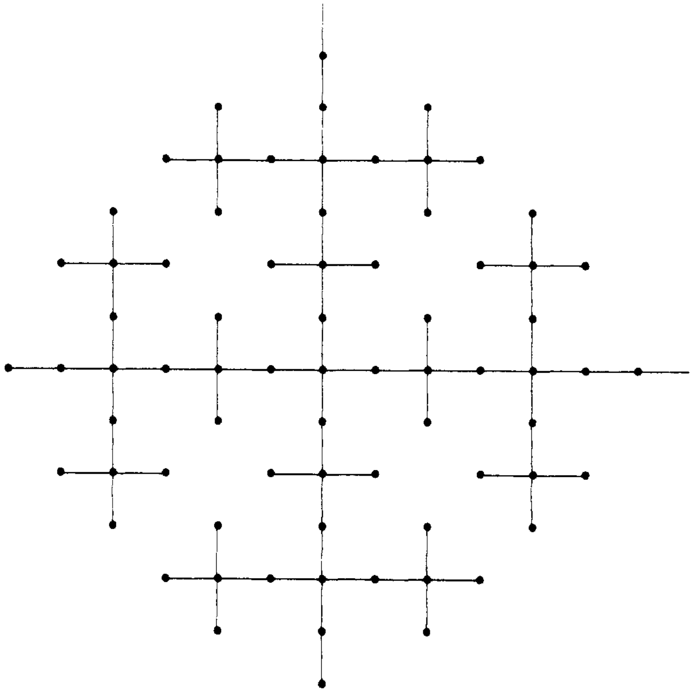

Figure 6. The generator of the generalized regular fractal model for DLA with $b = 13$ and $D_f = \ln 73/\ln 13$.

The above formula suggests that h_1 and h_2 are highly irrelevant, associated with the zero eigenvalue. The largest eigenvalue is given by $\tilde{D}_4$. The renormalization relation for the matrix $\tilde{t}$ has the form

$$D_4 A\tilde{t}' = t^{b-1}\tilde{t}. \tag{57}$$

At the critical point, equation (57), together with equation (54), provides

$$D_4 = b^{-f(q,\alpha)/2} \tag{58}$$

and thus

$$\omega_{\max} = \tilde{D}_4 = D_4^4. \tag{59}$$

Finally, we have

$$z = D_f + \frac{2f(q, \alpha)}{\nu}. \tag{60}$$

In comparison with KC, we note that the factor D_f in the dynamic critical exponent z results from the scaling of the most slowly relaxing perturbation, while $f(q, \alpha)/\nu$ comes from the scaling of the Liouville operator. On KC, the largest eigenvalue $\omega_{\max}$ corresponds to the scaling of the field parameter for a site with coordination number $r = 2$ (see section 3 and [9]), whereas on the regular fractal $\omega_{\max}$ corresponds to the scaling of the field parameter for a point with $r = 4$. This results in the different factors, 1 and 2, for KC and regular fractals, respectively, before $f(q, \alpha)/\nu$ in the expression of z. We conjecture that on 3D regular fractal models for DLA clusters, the dynamic exponent z should be

$$z = D_f + 3f(q, \alpha)/\nu \tag{61}$$

since the relevant field parameter corresponds to a site with $r = 6$. The calculation on some simple 3D regular fractals does show the scaling law of the dynamic critical exponent (61).

5. Conclusions and discussions

By means of TDRG theory for the kinetic spin model, we have studied the critical dynamics for some finitely ramified fractals and obtained the dynamic critical exponents. For all these fractals, the dynamic exponent z can be written in terms of the static correlation exponent ν. On the Koch curves, the scaling law of the dynamic exponent $z = D_f + f(q, \alpha)/\nu$, with α being the transition rate asymmetry coefficient, is found. On 2D regular fractal models for DLA clusters we have $z = D_f + 2f(q, \alpha)/\nu$. Furthermore, we note that no matter how complex these four classes of fractals may be, the perturbation of (6) always contributes D_f to the dynamic exponent z and the Liouville operator adds $f(q, \alpha)/\nu$ and $2f(q, \alpha)/\nu$ to z for Koch curves and regular fractals, respectively. The different factors before $f(q, \alpha)/\nu$ result from the fact that the relevant field parameter corresponds to sites with different coordination numbers on the two families of fractals. This leads us to conjecture that the 3D regular fractals should have the dynamic exponent $z = D_f + 3f(q, \alpha)/\nu$. This conjecture is supported by the calculation for some simple 3D regular fractals. Meanwhile, it is interesting to note that as in Ising model, the static exponent $\nu \equiv 1$ on the regular fractals, independent of the choice of the regular models, which seems to support the fact that the regular fractals can characterize the essential features of the random DLA clusters. Therefore, the dynamic exponent $z = D_f + 2f(q, \alpha)/\nu$ and $z = D_f + 3f(q, \alpha)/\nu$ are proposed for the kinetic Potts model on the 2D and 3D random DLA clusters, respectively.

Finally, we should note that in the context of this paper, we limit ourselves to a discussion of the critical dynamics on regular fractals. We tend to believe that the conclusion that the dynamic critical exponent depends only on the static properties is not only a consequence of the regular fractals, it also holds for other finitely ramified fractals, which have the zero critical temperature. Further, although the critical Glauber dynamics on fractal geometries is commonly discussed using Achiam's TDRG scheme [8, 9, 15], it is interesting to ask whether the dynamic exponent still depends only on statics if we go beyond the approximation. To answer this question one may work directly with the equations of motion for the expectation value of the spins [21]

$$\langle S_n(t)\rangle = \sum_{\{\sigma\}} \sigma_n P(\{\sigma\}; t)$$

instead of decimating at the level of the master equation. This has yet to be studied further.

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